DYNAMIC ARBITRAGE GAPS FOR FINANCIAL ASSETS

In a Non Linear and Chaotic Price Adjustment Process

Dr. Rodolfo Apreda

Professor at Del Cema University
Member of the Argentine Finance Association (AFA)
Member of the Eastern Financial Association (EFA), United States

Abstract

In this paper we are concerned with the existence of a dynamic arbitrage gap that evolves out of an adjustment process for disequilibrium prices, within a complex dynamics framework which takes into account the market microstructure and transactions costs.

Although this gap exhibits non linear and chaotic behavior, it doesn't preclude effective arbitrage transactions from taking place in real markets. Moreover, it may explain much better those factors which usually impede actual perfect arbitrage. Besides, this dynamic arbitrage gap depends upon a truly financial gap that accounts for unexpected events and superior information on the professional dealers´side.

In this way, we can learn much more about dynamical adjustment processes from financial assets, making the arbitrage gap instrumental to set about real arbitrage positions. Finally, the dynamic arbitrage gap could become useful when coping with financial crisis as far as some basic parameters´range of values for which the dynamics becomes chaotic could be measured in advance.

We acknowledge helpful comments and encouragement from Professors Edgardo Zablotsky and Ricardo Schefer, both in Del Cema University Faculty. Also, we are grateful to Chairwoman Professor Demmings, Miami University; and Discussant Professor Heusen, West Virginia University, for their valuable comments on a previous version of this paper presented at the Eastern Financial Association Meeting, April 1998, held at Williamsburg, Virginia, United States.

1.- INTRODUCTION

Market microstructure and non linear dynamics have aroused deserved expectations from both practitioners and academicians, for the last fifteen years. Moreover, not only the dealer's management of selling and buying orders and the specialist's activity, but the regulatory framework and contracting costs which seems to pervade markets as well, have become current issues in theory and practice. Furthermore, environmental signals and unexpected events have been given due attention in modelling price adjustments process. Not surprisingly, a growing financial literature has so far become less involved with the so called equilibrium prices convention, paying due attention to disequilibrium values, transaction prices and fundamental values. Outstanding research running along this path has been carried out by, among others, Cohen-Maier-Schwartz-Whitcomb (16) (17), Garbade-Silber (24), Benston-Smith (12), Blume-Siegel (13), Demsetz (22), Levy-Livingston (28), and mainly Beja-Goldman (9) or Beja (10),(11), and also Apreda (1) (2) (4) which are quoted in the References section. We are truly indebted to Beja's dynamic approach.

Getting onto the subject, our concern here is with Complex Dynamics. Briefly stated, economic dynamics is the systematic study of economic change. When the patterns of change can be translated by finite paths, convergent paths, or balanced settings, we deal with the so called simple dynamics. But if patterns of change can be translated by non-periodic paths, overlapping waves, switching regimes, or structural changes, then we face complex dynamics. Although research on this matter is a promising academic field, it seems to be only at the outset either in Financial or in Economic issues. Suitable references for mathematical foundations are found in Devaney (23), Hirsch-Smale (27) and Li-Yorke's paper (29). Seminal work has been carried out by Professor Day(19) (20) (21). Chaotic Dynamic Systems, and Chaotic Dynamic Processes applied to Finance are treated in Apreda (3) (4) (6); and a preliminary development of this paper's contents can be found in Apreda (1). Computation and Econometrics problems are dealt with in Brock (14). Recent work in non-linear dynamics to Finance is surveyed in Cuthbertson (18)

Before going into our proposal, we would like to stress an increasing empirical support to the following statements, drawn mainly from the foregoing sources:

- An adjustment process takes its time, and frequently is not convergent, and most of the time price sequences follow either stochastic processes or non linear complex behavior which includes chaotic itineraries within a range of fluctuation not necessarily wide or even fixed.

- An instantaneous adjustment can't really be afforded because of communication capabilities and rules of the trade in use. Adjustment processes in real markets evolve only at finite velocity, and allow for transactions to take place at market prices which are not necessarily equilibrium ones.

- Microstructure and transactions costs come up as an unavoidable trade-off with real markets, either when dealing with financial engineering or new regulation proposals, at setting up or improving financial institutions, in measuring financial performance, or helping markets to run more smoothly.

Briefly, what we want to do in this paper can be broken down into the following steps:

- Firstly, we introduce a transaction costs gap and a dynamic arbitrage gap.
- Next, a price adjustment model is built up by matching up two types of adjustment behavior: one produced by arbitrage, the other by speculators or well informed traders, both adjustment processes inclusive of transaction costs.
- Then, we present a dynamic setting with professional traders in the context of a market transactional structure.
- Finally, a complex dynamics will evolve that allows for disequilibrium stages, which we track by means of a dynamic arbitrage gap. This gap conveys to non linear dynamics and likely chaotic patterns. Furthermore, it is directly related to transactions costs and market microstructure.

2.- TRANSACTION COSTS GAP

Let us assume that the investment horizon is $\mathbf{[t\ ;\ T]}$. By $\mathbf{E[P_t\,(T)]}$ we mean the expected price at the moment T of a tradeable financial asset, assessed at time t. By $\mathbf{P_t\,(t)}$ we mean the current transaction price at the moment t, assessed at time t.

The wealth relative in the investment horizon comes up as

$$\mathbf{E\,[P_t\ (T)]\ /\ P_t\,(t) \quad = \quad 1 + \ E\,[\,r_t\,]}$$

where $\mathbf{E\,[\,r_t\,]}$ is the expected return through the horizon $\mathbf{[t\ ;\ T]}$.

We will denote k(t) the rate of transaction costs assessed at time t, for the moment t. By the same token, k(T) will denote the rate of transaction costs assessed at time t, for the moment T. That is to say, respectively, entrance and exit transaction costs. The wealth relative net of transaction costs follows from:

$$\mathbf{[\,1\,]} \qquad \mathbf{\{\,E\,[P_t\ (T)]\,.\,[1 - k(T)]\ \}\ /\ \{\,P_t\,(t)\,.\,[1 + k(t)]\,\} \quad = \quad 1 + \ E\,[\,r_{t,\,N}\,]}$$

where $\mathbf{E\,[\,r_{t,\,N}\,]}$ means the expected return, net of transaction costs. [Apreda (7)].

Now we can proceed along with a differential rate for the transaction costs:

$$[1 - k(T)] \; / \; [1 + k(t)] \; = \; 1 \; / \; [1 + k_t]$$

to get

$$[\,2\,] \qquad \{ E\, [P_t\;(T)] \,/\, P_t\,(t) \} \; = \; < 1 + \; E\,[\, r_{t,N}\,] >. \; < 1 + k_t >$$

Definition 1:

We call k_t , in [2] , the transaction costs gap.

Remarks:

- k_t is an all-inclusive transaction costs gap.
- we could have done $[1 - k(T)] \; / \; [1 + k(t)] \; = \; [1 + g_t]$ for instance, but we picked up the other way so as to factor in the nominal return in terms of the net return and the transaction costs gap.

3.- ARBITRAGE GAP

By $E[\, rw_t\,]$, we mean the expected rate of return from any financial asset, assessed at the moment "t" for period **[t ; T]** in an ideal market, such as it is provided by a qualifying valuation model.

Taking into account transaction costs, it is worth trying arbitrage whenever the discrepancy between the theoretical price and the current price covers up transaction costs and leaves something to take advantage of. To see this, let us make

$$[\,3\,] \qquad < 1 + \; E\,[rw_t\,] > . < 1 + k_t > . < 1 + b_t > \; = \; < 1 + E\,[r_t\,] >$$

In this equation, b_t performs as a differential rate. It measures the expected discrepancy between the tradeable asset and the modelled one, once transaction costs are covered. Several applications of this rate can be found in Apreda (5) (6) or in Marshall (29). This remark brings about the following

Definition 2

We will call b_t, in [3], the arbitrage gap.

This rate, when being different from zero, triggers off plausible non-zero arbitrage positions. Still, these positions don't need to be attainable ones, unless costs of transaction may be covered and microstructure features could be coped with, as we are going to see in the following paragraphs.

Remarks :

- The compatibility between definitions 1 and 2 follows from:

$$<1 + E[rw_t] > . <1 + k_t > . <1 + b_t> = <1 + E[r_t]> = <1 + E[r_{t,N}]>. <1 + k_t >$$

and then,

$$<1 + E[rw_t] > . <1 + b_t> = <1 + E[r_{t,N}]>$$

- $E[rw_t]$ doesn't convey transaction costs in most valuation models. In this sense, is "net" of transaction costs.

4.- DYNAMIC ARBITRAGE GAP

We are going to link effective returns to continuous returns and prices along (t ;T). Current spot and expected futures prices assessed at "t" will be denoted by $\mathbf{p_t\ (t)}$ and $\mathbf{E[p_t\ (T)]}$, respectively. On the other hand, model or "equilibrium" prices assessed at "t" will be denoted by $\mathbf{w_t\ (t)}$ and $\mathbf{E[w_t\ (T)]}$, respectively. For "r_t" we will understand the tradeable financial asset expected continuous return from the tradeable financial asset, and for "rw_t" the modelled financial asset expected continuous return from the modelled financial asset. So we get:

$$\exp[r_t . (T - t)] = < 1 + E [r_t] > = E [p_t (T)] / p_t (t)$$

$$\exp[rw_t . (T - t)] = < 1 + E [rw_t] > = E [w_t (T)] / w_t (t)$$

From these relations and **[3]** it follows:

$$\ln<E [w_t (T)] / w_t (t)> + \ln<1+ k_t > + \ln<1+ b_t > = \ln<E [p_t (T)] / p_t (t) >$$

If we put

$$\mathbf{< 1 + k_t > \; = \; exp\,[\, - K(t)\,(T - t)]}$$

$$\mathbf{< 1 + b_t > \; = \; exp[\, - \beta(t)\,.\,(T - t\,)]}$$

we arrive at

$$\mathbf{- \beta(t)\,.\,(T - t\,) \quad = ln{<}E\,[p_t\,(T)] \,/\, p_t\,(t) > \; - ln{<}E\,[w_t\,(T)] \,/\, w_t\,(t) > \; - \; ln{<}1{+}\,k_t >}$$

But the expected future price from the traded asset should likely be assessed by the same model used to assess the future price of the ideal asset. That is to say:

$$\mathbf{E\,[\,p_t\,(T)\,] \;\; = E\,[\,w_t\,(T)\,]}$$

Hence:

$$\mathbf{- \; \beta(t) \quad = \; \{\,[\,ln\,w_t\,(t) - ln\,p_t\,(t)\,] \,/\, \; (\,T - t\,)\,\} \; + \; K(t)}$$

Let us denote, as usual,

$$\mathbf{ln\,w_t\,(t) \; = \; W(t) \qquad ; \qquad ln\,p_T(t) \; = \; P(t)}$$

and we get:

$$\mathbf{[\,4\,] \qquad\qquad \beta(t) \quad = \; \{\,[\,P\,(t) - W(t)\,] \,/\, \; (\,T - t\,)\,\} \; - \; K(t)}$$

<u>Remarks:</u>

a) By W(t) = W_t (t) we mean the fundamental value assessed at “t” for moment “t”, brought about by a valuation model.

b) We put

$$\mathbf{< 1 + b_t > \;\; = \; exp[\, - \beta(t)\,.\,(T - t\,)]}$$

bearing in mind the gap will narrow as far as the market arbitrages evolves trying to close the gap. Indeed, our dynamic approach may shed light onto Goldman-Sosin’s measure of inefficiency in a market [Goldman-Sosin (26)]

$$\mathbf{MI \;\; = \; E\,[\,(\,ln\,p^f{}_t - ln\,p^p{}_t\,)^{\,2}\,]}$$

where p^f is a price in a fully informed economy and p^p stands for a price in a partially informed economy. We are going to make a new reference to this issue (although departing from this setting towards an arbitrage approach) in part 6 which will be devoted to price adjustment.

c) We put

$$< 1 + k_t > \;=\; \exp[\, - K(t) \, . \, (T - t)\,]$$

just to get W(t) and K(t) with the same sign in [4]

The former analysis leads to a new definition.

<u>Definition 3:</u>

By dynamic arbitrage gap we mean

$$\boldsymbol{b}(t) \;=\; [\, 1 / (T - t)\,] \, . \, [\, P(t) - W(t)\,] \; - \; K(t)$$

We see that the dynamic arbitrage gap defines a temporal path which depends on traded prices, model values , and transaction costs through the investment period. For a unit period horizon:

$$\beta(t) \;=\; P(t) \; - \; W(t) \; - K(t) \qquad [\,5\,]$$

<u>Remark:</u>

In terms of continuously compounded rates this amounts to:

$$\beta(t) \;= r_t \; - \; rw_t \; + \; k_t$$

5.- MODELLING THE DYNAMIC ARBITRAGE GAP

Two different approaches stand a chance of modelling the dynamic arbitrage gap behavior: the deterministic and the stochastic ones. We deal with the deterministic approach through this paper, and have coped with the stochastic model elsewhere; on this account, see Apreda (2) (6) (8).

We will take the following steps:

a) adoption of a deterministic dynamical model for price adjustment behavior;

b) framing a dynamic setting with professional traders and market transactional structure;

c) disequilibrium prices, embedded in a complex dynamics framework, will enable us to prove two lemmas on the existence of a dynamic gap, its relationship with the market microstructure and transaction costs, showing that such an arbitrage gap also conveys a likely chaotic behavior.

6.- PRICE ADJUSTMENT BEHAVIOR

It is worth distinguishing, for our purposes, between an economy which is adequately informed and another economy which is only ineficciently informed.

An economy is meant to be adequately informed whenever:

- Agents reach their decisions by means of bounded rationality.
- Information flows allow agents for successful arbitrages, most of the time.

Remarks:

a) In this context, for arbitrage to take place it should be required, from [5] :

$$\boldsymbol{\beta(t) = P(t) - W(t) - K(t) \rightarrow 0}$$

b) Beja (9) compares an equilibrium price with the current price, and Goldan-Sossin (26) compare a price in a completely informed economy with a price in a partially informed economy. We depart from these approaches because we feel more sensible to take an "adequately" informed economy, that is to say, an economy where there is no serious hindrance to arbitrage, as a proxy of equilibrium or perfect information in real markets.

c) Bounded rationality is used in Herbert Simon's sense (as found in his work "Administrative Behavior", The Free Press, fourth edition, 1997). A very interesting economic development of this concept was made by Reiter for markets out of equilibrium . Reiter (31).

An economy is meant to be inefficiently informed whenever

- Agents reach their decisions by means of bounded rationality.
- Market microstructure hampers transactions, because of lags, breadth and depth market, regulations, contracting and transaction costs.

- There are agents with superior information ("superior traders") who claim competitive advantages to play in the market. Cuthbertson in (18) surveys relevant research on this behavior.

Changes in prices have a primary source in excess demand

$$\varepsilon(P(t)) = D(P(t)) - S(P(t))$$

and price dynamics, in a continuous setting, may run as

$$[6] \qquad dP(t)/dt = H(\varepsilon(P(t))$$

where H stands for a increasing monotonous function which is null at t = 0.

Allowing for the fact that economic agents really carry over their buying and selling at transaction prices instead of theoretical ones (such as those provided by qualifying valuation models), with imperfect adjustment mechanisms, we can write:

$$[7] \qquad \varepsilon(P(t)) = \varepsilon(f, P(t)) + \varepsilon(\Delta, P(t))$$

where the excess of demand splits up into two components, the first of them taking the place of that excess demand which would follow if the market belonged to an adequately informed economy. That is why, within such an economy,

$$\varepsilon(f, P(t))$$

this excess of demand should be called fundamental. The second component

$$\varepsilon(\Delta, P(t))$$

measures the gap between the effective excess demand and the fundamental one.

Which price would be a fair one for the fundamental demand? We will denote it by means of

$$W(t) + K(t)$$

And this amounts to

$$\varepsilon(f, W(t) + K(t)) = 0$$

Let us consider the following adjusting dynamics

$$[\,8\,] \qquad \boldsymbol{\varepsilon}(\mathbf{f}, \mathbf{P(t)}) = \mathbf{g_1}\,[\,\mathbf{P(t)} - \mathbf{W(t)} - \mathbf{K(t)}\,]$$

where g_1 is an increasing monotonous function. This price adjustment takes into account transaction costs. Not surprisingly, within square brackets we find the dynamic arbitrage gap.

Remarks:

a) Although most of the markets participants behave as "smart money" or rational agents, recent academic research has included a subset of "irrational" or "noise" traders who don't quote prices equal to fundamental values, and still survive in the market, in spite of arbitrage (See Cuthbertson (18)). Allowing for this "heterogenous traders", the rational agents must take into account what the former would do in the future and behave accordingly.

b) At this moment, it is worth recalling what Stahl and Fisher wrote in their work about stability with disequilibrium awareness (31): "Agents in Walrasian world formulate demands again and again taking prices as given and paying no attention to the fact that they will often not be able to complete their planned transaction".

The driving force for price changes follows from **[7]** and **[8]**

$$\boldsymbol{\varepsilon}(\Delta, \mathbf{P(t)}) = \boldsymbol{\varepsilon}(\mathbf{P(t)}) - \mathbf{g_1}\,[\,\mathbf{P(t)} - \mathbf{W(t)} - \mathbf{K(t)}\,]$$

Whereas equilibrium excess demand shows positions wished by agents if prices were those attainable by arbitrage, $\boldsymbol{\varepsilon}(\Delta, \mathbf{P(t)})$ is an excess demand that takes place only when disequilibrium comes up to breed speculation, at transaction prices. It is an environmental excess demand. Here we are taking advantage of Beja (8).

It is this disequilibrium which prompts the demand for the financial asset on the grounds of future price ex0pectations, **but also push up demand for other alternative assets**. Hence, this addtional excess demand can't be explained by the fundamental one.

An unexpected event, as the arrival of new information, will bring about an intertemporal sequence of price changes, matching the sequence of excess demand variations which come up as long as that information is spread over among the market participants. It is an striking fact that the process not only can be stochastic [Apreda (2),(6)] but non linear and chaotic, as well .

Now, we are ready to blend in this additional excess demand into an adjustment adaptive process. In order to do that, let us introduce two new variables:

- The speculator's forecast about the continuous return trend from the asset, in terms of a rate net of transaction costs, $\mathbf{r_{sp}\ (t)}$, plus the continuous rate of transaction costs $\mathbf{K(t)}$

$$\mathbf{r_{sp}\ (t)\ +\ K(t)}$$

- A market0 index continuous rate of return, taken as a benchmark for financial opportunity costs.

$$\mathbf{q(t)}$$

Remarks:

a) Opportunities to trade on different terms are a feature of markets out of equilibrium as Reiter pointed out in (31), adding that information's role has two main aspects: first, the institutional structure of the market determines the information that agents may get from the market process, which can be called the structural aspect. Second, the restricted capacity of economic agents to handle information, and this can be called the bounded rationality aspect. By the way, that's why some agents would have incentives to become mediators in such a market.

b) The speculator's forecast doesn't necessarily coincides with E [r(t)], as assessed by a general agent, or an arbitrageur, as long as the speculator takes advantage of superior information and the market microstructure. A telling portfolio management approach to superior information within a transactions costs setting can be found in Levy-Livingston (27).

c) Capelin and Leahy (14) made the following daring remarks in an article published in May 1996:

- " Frictions prevent trade and therefore impede the acquisition of information. There is no reason to view the price as a sufficient statistic for the state of the market. The volume of trade will provide important additional information."

- "Trading costs provide an explanation for the commonly observed combination of sharp contractions and slow expansions; they provide an explanation for downward price rigidity. It is a mistery that prices have so little explanatory power in many markets."

d) Cuthbertson's book (18) is very useful to address borderline research in Finance.

The foregoing adaptive adjustment process now can be given by:

$$\boldsymbol{\varepsilon(\ \Delta,\ P(t)\ )\ =\ g_2\ [\ r_{sp}\ (t)\ +\ K(t)\ -\ q(t)\ ]}$$

where $\mathbf{g_2}$ is increasing monotonic which is null at t = 0. Furthermore, it signals the agent awareness of disequilibrium which is brought about by incoming new events.

The foregoing statements enable us to plug these relations into **[3]** and set up a dynamics:

$$[\,9\,] \quad dP(t)/dt = H\{g_1\,[\,P(t) - W(t) - K(t)\,] + g_2\,[\,r_{sp}(t) + K(t) - q(t)\,]\,\}$$

Assuming standard regularity conditions, developing the function H in Taylor Series and picking up the first order approximation:

$$[\,10\,] \quad dP(t)/dt = a \cdot [\,P(t) - W(t) - K(t)\,] + b \cdot [r_{sp}(t) + K(t) - q(t)\,] + o(dt)/dt$$

This amounts to an approximating discrete model

$$[\,11\,] \quad dP(t)/dt \approx \Delta P(t)\,/\,\Delta t = a \cdot [\,P(t) - W(t) - K(t)\,] + b \cdot [\,r_{sp}(t) + K(t) - q(t)\,]$$

7.- DYNAMIC SETTING WITH TRANSACTIONAL STRUCTURE AND MEDIATOR

A mediator, or professional trader, buys to suppliers financial assets at a price

$$p(t)$$

and sells marking up at a rate of "ν" per unit, that is

$$(1 + \nu) \cdot p(t)$$

On the other hand, the excess demand

$$\varepsilon(\,p(t)\,) = D(\,p(t)\,) - S(\,p(t)\,)$$

sensitizes itself to supply and demand shifts, calling for a parametric pattern as

$$\varepsilon(\,p(t), \mu\,) = \mu \cdot \varepsilon(\,p(t)\,) = \mu \cdot [\,D(\,p(t)\,) - S(\,p(t)\,)\,]$$

taking any deviation from the benchmark $\mu = 1$, as a market's breadth measure; see Garbade (23). This embeds the excess demand in a market microstructure setting.

Adopting a simple relationship between price changes and the excess demand, Samuelson-sort, it follows:

$$\mathbf{p(t+1) - p(t) = g [\varepsilon(p(t), \mu)] \quad ; \quad g' > 0}$$

claiming that the increasing monotonic function is given by

$$\mathbf{g [p(t)] = \lambda \cdot \varepsilon(p(t), \mu) \quad ; \quad \lambda > 0}$$

where λ can be regarded as an adjustment price velocity.

Remarks:

The determination of λ lays on these grounds:

a) Firstly, the increase in the supply of the financial asset translates an inventory change:

$$\mathbf{s(t+1) - s(t) = - \varepsilon(p(t), \mu, \nu) = - \mu \cdot \{ D [(1+\nu) p(t)] - S(p(t)) \}}$$

b) In real life, the mediator doesn't know neither the supply-demand structure, nor the clearing price. The relevant variable here, for him, is how his own inventory changes allowing him to mark up or down his transaction price; see Day (17). The adjustment comes up as:

$$\mathbf{p(t+1) - p(t) = \lambda \cdot [s(t+1) - s(t)]}$$

Now he can estimate λ, as prices change and adjustement through excess demand takes place,

$$\mathbf{p(t+1) - p(t) = \lambda \cdot \varepsilon(p(t) , \mu) \quad ; \quad \lambda > 0}$$

that we can rewrite as

$$\mathbf{p(t+1) - p(t) = \lambda \cdot \mu \cdot [D (p(t)) - S(p(t))]}$$

c) Lastly, if we take into account the mediator spread:

$$\mathbf{[\ 13\ ] \quad p(t+1) - p(t) = \lambda \cdot \mu \cdot [D [(1+\nu) p(t)] - S(p(t))]}$$

As a consequence of **[13]**, changes in prices come from excess demand with respect to spread structure, adjustment velocity in terms of excess demand and, finally, the breadth of the market. This embeds the price change in a market microstructure setting and takes into account transaction costs directly related to the mediator.

d) Preventing negative prices, we arrive at the expected price as function of the breadth of the market, adjustment velocity, mark-up and excess demand:

$$[\ 14\ ] \quad \mathbf{p(t+1) = \theta(p(t)) = Max\ \{\ 0,\ p(t) + \mu \cdot \lambda \cdot \{\ D\ [\ (1+\nu)\ p(t)\ ]\ -\ S(\ p(t)\ )\ \}\}}$$

$\boldsymbol{\theta(p(t))}$ determines an orbit as long as "t" evolves, for each initial condition $p(t_0)$, and the mapping can be considered a dynamical system. [Apreda (3)] [Hirsch, Smale (27)]

<u>7.1.- STABILITY ANALYSIS</u>

It has been proved by Day (19) that stability analysis applied to relation **[14]** leads to the following strong statement:

The relative adjustement in normal markets shows every feature attending simple dynamics and complex dynamics. In general, convergence towards an only competitive equilibrium, convergence to periodic cycles, chaotic topology, strongly chaotic trajectories and, at last, self-destruction.

Further, it has also been proved that for normal demand and supply functions, both mediator tatonnement and relative mediator tatonnement displays the following features:

- a unique nonstationary state p^o exists under competitive settings;
- for a wide range of parameters values, the process converge to a market clearing equilibrium;
- for a wide range of parameter values, the process exhibits cyclic or chaotic price sequences.

The reader can find the stability and relative stability analysis fully expanded in the Appendix (see section 9)

8.- COMPLEX DYNAMICS WITH DISEQUILIBRIUM PRICES

After encompassing both the dynamic adjustment behavior and the complex-dynamics with professional traders, we are ready for dealing with the following lemma.

<u>Lemma 1</u>

Within the framework of the dynamical model for prices introduced in part 6, and the complex dynamics with mediator developed in part 7, it follows:

$$[15]\quad Max\, \{ -1 \; ; \; [\, l \,.\, e(\, p(t)\,)\,] \;\text{¸}\; [\, Max\, \{D((1+n)\, p(t)\,) \; ; \; S(\, p(t)\,)\,\}\,]\,\}$$

$$= a \cdot [\, P(t) - W(t) - K(t)\,]\, Dt \;+\; b \cdot [\, r_{sp}(t) + K(t) - q(t)\,]\; Dt$$

Proof:

We will go on through steps.

<u>Step 1</u>: In a discrete and deterministic framework, the total return is translated by:

$$\mathbf{r(t, t+1) \;=\; \Delta\, p(t) \;/\; p(t)}$$

Applying the relative adjustment we left to develop in the Appendix (paragraph 9.3) it follows:

$$\mathbf{r\,(t, t+1) \;=\; [p(t+1) - p(t)] \;/\; p(t) \;=}$$

$$\mathbf{Max\; \{ -1 \; ; \; [\, \lambda \,.\, \varepsilon(\, p(t)\,)\,] \;\div\; [\, Max\; \{\, D((1+\nu)\, p(t)\,) \; ; \; S(\, p(t)\,)\,\}\,]\; \}}$$

<u>Step 2</u> : Providing standard regularity conditions are fulfilled, as we saw in part 2, we can take advantage of **[12]**

$$\mathbf{\Delta\, p(t) \;/\; \Delta t \;=\; a \cdot [\, P(t) - W(t) - K(t)\,] \;+\; b \cdot [\, r_{sp}(t) + K(t) - q(t)\,]}$$

Recalling that

$$\mathbf{P(t) \;=\; ln\; p(t)}$$

it follows

$$\mathbf{\Delta\, p(t) \;/\; p(t) \;=\; a \cdot [\, P(t) - W(t) - K(t)\,]\, \Delta t \;+\; b \cdot [\, r_{sp}(t) + K(t) - q(t)\,]\; \Delta t}$$

Furthermore,

$$\mathbf{Max\; \{ -1 \; ; \; [\, \lambda \,.\, \varepsilon(\, p(t)\,)\,] \;\div\; [\, Max\; \{\, D((1+\nu)\, p(t)\,) \; ; \; S(\, p(t)\,)\,\}\,]\; \} \;=}$$

$$\mathbf{a \cdot [\ P(t)\ -\ W(t)\ -K(t)\ ]\ \Delta t\ \ +\ \ b \cdot [\ r_{sp}\,(t)\ +K(t)\ -q(t)\ ]\ \ \Delta t}\quad \ddot{y}$$

Now, if we recall [4] , then the expression,

$$\mathbf{W(t)\ -\ P(t)\ -K(t)}$$

is, by no means, the dynamic arbitrage gap.

However, the nature of this gap is far from simple. In the framework of a complex deterministic dynamics, it matches current prices trajectories against those of the fundamental prices. But these trajectories can be periodical or chaotical. [In the framework of stochastic process it can be regarded as a brownian one, as proved in Apreda (2) (6).]

Coming back to the departure relation and substituting the dynamic arbitrage dynamic gap for **W(t) - P(t) - K(t)** , and taking unit intervals:

$$\mathbf{Max\ \{\ -1\ ;\ [\ \lambda\ .\ \varepsilon(\ p(t)\ )\ ]\ \div\ [\ Max\ \{\ D((1+\nu)\ p(t)\ )\ ;\ S(\ p(t)\ )\ \}\ ]\ \}\ =}$$

$$\mathbf{a \cdot \beta(t)\ \ +\ \ b \cdot [\ r_{sp}\,(t)\ +K(t)\ -q(t)\ ]}$$

Now we are ready to prove an existence lemma.

<u>Lemma 2</u>

Let us take a financial asset in a market with mediator. Then, there is a dynamic arbitrage gap between the model value and the transaction price, which locally depends on the market structure, the transaction costs and a financial gap.

Proof:

From the relation **[15]** above, it holds that

$$\mathbf{Max\ \{\ -1\ ;\ [\ \lambda\ .\ \varepsilon(\ p(t)\ )\ ]\ \div\ [\ Max\ \{\ D((1+\nu)\ p(t)\ )\ ;\ S(\ p(t)\ )\ \}\ ]\ \}\ =}$$

$$\mathbf{a \cdot \beta(t) \quad + \; b \cdot [\; r_{sp}\,(t) \; + K(t) \; - q(t)\;]}$$

The left hand expression stands for the transactional structure of the market through the parameter λ, the professional trader mark-up, and the excess demand. The expression

$$\mathbf{b \cdot [\; r_{sp}\,(t) \; + K(t) \; - q(t)\;]}$$

should be regarded as a financial gap between the asset forecast return and the oportunity cost from alternative investments in the market.

By isolating the gap:

$$\mathbf{\beta(t) \;=\; Max\{ (-1/\,a\;);[(\lambda/a\,)\,\varepsilon(\,p(t)\,)\,] \div [Max\,\{D((1+\nu)\,p(t)\,)\,;S(\,p(t)\,)\}]\;\}}$$

$$\mathbf{-\;(b\,/\,a)\,.\,[\;r_{sp}\,(t)\;+K(t)\;-q(t)\,]}$$

<u>Remark:</u>

If we raised the question whether we can track the arbitrage gap along stochastic trajectories, the answer would be affirmative, as we have proved in other research paper, making use of stochastic differential equations; see Apreda (2),(6).

7.1.- CONSEQUENCES

- Changes in the parameters are the most usual feature for a model to explain what is going on in real markets.

- It is a foremost consequence from non linear dynamics that changes in parameters value give rise to bifurcations periodic trajectories and, for certain range of values, chaotic trajectories.

- The presence of chaos doesn't convey the idea of non tractability. On the contrary, there are analytical tools that may lead to the range of parameter values which take to chaotic paths. (See Cuthbertson (18) or Day (20) (21))

- By the foregoing remarks, simulation models could help us into the management of the dynamic arbitrage gap in the range of parameters values that lead to chaos.

- We hope this model to shift conventional wisdom ground on how to understand much better real financial markets and to cope with financial crises.

8.- CONCLUSIONS

a) We have produced a dynamic model for disequilibrium prices behavior closely intertwined with the market microstructure.

b) In this context, it has been proved the existence of a dynamic arbitrage gap.

c) The dynamic arbitrage gap is brought about with an explicit relationship among the market transactional microstructure, the transaction costs system and a financial gap.

d) An outstanding conclusion regarding the dynamic arbitrage gap could be stated as follows:

- Price trajectories patterned with market microstructure, transactional costs and a financial gap makes rather uncommon that there might be convergence towards a specific and only value as from which the arbitrage is not more feasible.

- On the other hand, when there are no more arbitrage opportunities because not even transaction costs are broken even, this doesn't mean that convergence towards equilibrium has been reached, as "zero-arbitrage" model suppose, because the trajectory of observable prices could be chaotic, in spite of the gap width.

e) Modelling the dynamic arbitrage gap as a deterministic system seems to be more realistic so as to understand the financial assets price mechanism in a deeper way. Besides, the quasi-stochastic behavior in prices that comes out of the chaotic trajectories, should make the model suitable for simulation trials, and this is a promising field for applied research..

9.- APPENDIX

9.1.-LOCAL STABILITY:

Recalling **[8]** :

$$\mathbf{p(t+1) = \theta(p(t)) = Max\ \{ 0,\ p(t) + \mu \cdot \lambda \cdot \{ D\ [\ (1+\nu)\ p(t)\ ] - S(\ p(t)\ )\ \}\}}$$

Taking derivatives with respect to price:

$$\theta'(p(t)) = 1 + \mu \cdot \lambda \cdot \{ (1+\nu) \cdot D'[(1+\nu)\, p(t)] - S'(p(t)) \}$$

As Richard Day (17) (18) has proved, based on Li -Yorke seminal article (28), the most remarkable stability characteristics are:

There is only one stationary state.

There is a range of parameter values that grants convergence towards equilibrium.

There is a range of parameter values where the dynamic process shows price trajectories which are cyclical or chaotical.

9.2.- RATES OF RETURN ADJUSTMENT

Price dynamics has been introduced as

$$p(t+1) = p(t) + g[\varepsilon(p(t)]$$

where "g" is an increasing monotonic function of the excess demand.

Let us take the function:

$$g[\varepsilon(p(t)] = < \mu\cdot\lambda \cdot \varepsilon(p(t)> \div < \mathrm{Max}\{ \mu D[(1+\nu)\, p(t)] ; \mu S[p(t)] \} >$$

which is increasing monotonic and allows to treat the dynamics in terms of rates of return:

$$r(t, t+1) = [p(t+1) - p(t)] / p(t) =$$

$$[\mu\cdot\lambda\cdot\{D[(1+\nu)\, p(t)] - S(p(t))\}] / \mathrm{Max}\{\mu D[(1+\nu)\, p(t)] ; \mu S(p(t))\}$$

We have translated the porcentual change in prices by a multiple of the excess demand, in terms of the long position of the market (the greater of demand or supply as the long side). Now we are ready to face the difference equation that holds in the price adjustment. Firstly, as

$$p(t+1) / p(t) =$$

$$1 + [\mu\cdot\lambda\cdot\{D[(1+\nu)\, p(t)] - S(p(t))\}] / \mathrm{Max}\{\mu D[(1+\nu)\, p(t)] ; \mu S(p(t))\}$$

we can proceed to

$$p(t+1) =$$

$$\mathrm{Max}\{\ 0, p(t) + p(t) \,.\, [\lambda \cdot \{D[(1+\nu)p(t)] - S(p(t))\}] \div \mathrm{Max}\{D[(1+\nu)p(t)]\ ;\ S(p(t))\ \}$$

9.3.- STABILITY IN RELATIVE ADJUSTMENT

Adopting normal supply and demand behaviour, it follows:

$$\theta(p(t)) \quad =$$

$$\mathrm{Max}\{\ 0, p(t) + p(t) \,.\, [\lambda \,.\, \{D[(1+\nu)p(t)] - S(p(t))\}] \div \mathrm{Max}\{D[(1+\nu)p(t)]\ ;\ S(p(t))\ \}$$

The function splits itself into two branches as long as the observable prices be less or bigger than the fundamental (equilibrium) price p(ê).

a) if observable prices are less than p(ê), then the demand is bigger than the supply:

$$\theta(p(t)) \quad = p(t) + p(t) \,.\, [\lambda \,.\, \{D[(1+\nu)p(t)] - S(p(t))\}] \div \{D[(1+\nu)p(t)]$$

$$\theta(p(t)) \quad = p(t) + p(t) \,.\, \lambda \ - \ \lambda \,.\, p(t) \,.\, \{\ S(p(t)) \div D[(1+\nu)p(t)]\ \}$$

$$\theta(p(t)) \quad = p(t) \,.\, (\ 1\ +\ \lambda\ )\ -\ \lambda \,.\, p(t) \,.\, \{\ S(p(t)) \div D[(1+\nu)p(t)]\ \}$$

b) if observable prices are bigger than p(ê), then the demand is less than the supply:

$$\theta(p(t)) \quad = \quad \mathrm{Max}\{\ 0, p(t) + p(t) \,.\, \{\ [\lambda \,.\, \{D[(1+\nu)p(t)] - S(p(t))\}] \div\ S(p(t))\ \}$$

$$\theta(p(t)) \quad = \quad \mathrm{Max}\{\ 0, p(t)\ -\ p(t) \,.\, \lambda \ +\ \lambda \,.\, p(t) \,.\{\ D[(1+\nu)p(t)] \div\ S(p(t))\ \}$$

$$\theta(p(t)) \quad = \quad \mathrm{Max}\{\ 0, p(t) \,.\, (\ 1 - \lambda\ )\ +\ \lambda \,.\, p(t) \,.\{\ D[(1+\nu)p(t)] \div\ S(p(t))\ \}$$

Stability, in a) excludes negative prices which could attend b) is look for through:

$$\theta'(p(t)) \quad = \quad 1 + \lambda \ -\ \lambda \,.\, \{\ \ S(p(t)) \div D[(1+\nu)p(t)]\ \ \}\ .$$

$$\{\ 1 + < [S'(p(t)).p(t)] \div S(p(t))> - <\{D'[(1+\nu)p(t)].(1+\nu).p(t)\} \div D[(1+\nu)p(t)]> \quad \}$$

but there are elasticities within this expression. Hence:

$$\theta'(p(t)) = 1 + \lambda - \lambda . \{S(p(t)) \div D[(1+\nu)p(t)]\} . \{1 + \eta^{S}(p(t)) - \eta^{D}(p(t))(1+\nu)\}$$

10.- REFERENCES

1.- Apreda, Rodolfo
Complex Financial Dynamics for Arbitrage Gaps
Paper presented at the 34 Annual Meeting, Eastern Financial Association
Williamsburg, Virginia, April 1994

2.- A Stochastic Dynamic Arbitrage Gap Model
(with transaction costs and market microstructure included)
Research Paper, Department of Finance, UADE, Buenos Aires, 1996.

3.- Apreda, Rodolfo
Chaotic Dynamic Systems
Research Paper, Department of Finance, UADE, Buenos Aires, 1997

4.- Apreda, Rodolfo
Arbitrage Dynamic Gap: a sufficient condition grantin chaos
Research Paper, Department of Finance, UADE, forthcoming

5.- Apreda, Rodolfo
Budgetary Gaps in Corporate Finance
The Journal of Financial Executives, IAEF, Buenos Aires, 1992

6.- Apreda, Rodolfo
Financial Costs, Dynamic Gaps and Arbitrage
PhD Dissertation, University of Buenos Aires, Buenos Aires, 1996

7- Apreda, Rodolfo
Budgetary Gaps and All-in-Cost Gaps
Research Paper, Department of Finance, UADE, Buenos Aires, 1995

8.- Apreda Rodolfo
Arbitrage Dynamic Gap viewed as an Stochastic Process inclusive of transaction costs
Del Cema University, Working Papers, forthcoming 1999

9.- Beja, Avraham; Goldman, Barry
On the Dynamic Behavior of Prices in Disequilibrium
The Journal of Finance, vol 35, Nº2, May 1980

10.- Beja, Avraham; Goldman, Barry
Market Prices vs. Equilibrium Prices: Return's Variance, Serial Correlation
and the Role of the Specialist
The Journal of Finance, Vol.34, Nº 3,Junte 1979

11.- Beja, Avraham; Hakansson
Dynamic Market Processes and the Rewards to up-to-date information
The Journal of Finance, Vol.32, Nº 2, May 1977

12.- Benston, George; Smith, Clifford
A Transactions Cost Approach to the Theory of Financial Intermediation
The Journal of Finance, Vol.XXXI, May, 1976

13.- Blume, Marshall; Siegel, Jeremy
The Theory of Security Pricing and Market Structure
Financial Markets, Institutions and Instruments; Volume 1; Number 3
New York University, Salomon Center, 1992

14.- Brock, William; Hsieh, David; LeBaron, Blake
Nonlinear Dynamics, Chaos, and Instability. (Statistical Theory and Economic Evidence)
The Mit Press, Massachusetts, 1993

15.- Capelin, Andrew; Leahy, John
Trading Costs, Price and Volume in Asset Markets
The American Economic Review, Vol. 86, Nº 2, May 1996

16.- Cohen, Kalman; Hawawini, Gabriel; Maier, Steven;Schwartz, Robert; Whitcomb, David
Implications of Microstructure Theory for Empirical Research on Stock Price Behavior
The Journal of Finance, Vol. XXXV, Nº 2 , May 1980

17.- Cohen, Kalman; Maier, Steven; Schwartz, Robert; Whitcomb, David
The Returns Generation Process, Return Variance,
and the Effect of Thinnes in Securities Markets
The Journal of Finance, Volume 33, Nº 1, March 1978

18.- Cuthbertson, Keith
Quantitative Financial Economics
Jon Wiley, New York, 1996
19.- Day, Richard
Complex Economic Dynamics
Volume 1, The Mit Press, Massachusetts, Usa, 1994

20.- Day, Richard
Complex Economic Dynamics:
Obvious in History, Generic in Theory, Elusive in Data
Nonlinear Dynamics, Chaos and Econometrics
Hashem Pesaram and Simon Potter. John Wiley, New York, 1993

21.- Day, Richard
Multiple-Phase Economic Dynamics
Nonlinear and Convex Analysis in Economic Theory
(edited by Toru Maruyama and Wataru Takahashi) ; Springer, New York, 1995

22.- Demsetz, H
The Cost of Transacting
Quarterly Journal of Economics, 1968, Volume 82; Number 1; pp 33-53

23.- Devaney, Robert
An Introduction to Chaotic Dynamical Systems

Addison-Wesley, New York, 1989

24.- Garbade, Kenneth; Silber, William
Structural Organization of Secondary Markets
The Journal of Finance, vol XXXIV, Nº 3, June 1979

25.- Garbade, Kenneth
Securities Markets
McGraw-Hill, New York, 1988

26.- Goldman, Barry; Sosin, Howard
Information Dissemination, Market Efficiency and the Frequency of Transactions
Journal of Financial Economics, Vol 7, pp 29-61

27.- Hirsch, Morris; Smale, Stephen
Differential Equations, Dynamical Systems, and Linear Algebra
Academic Press, New York, 1974

28.- Levy, Azriel; Livingston, Miles
The Gains from Diversification Reconsidered: Transaction Costs and Superior Information
Financial Markets, Institutions and Instruments, Vol.4, Nº 3, 1995

29.- Li, Then-Yien and Yorke, James
Period Three implies Chaos
American Mathematical Monthly, Vol.82, pp. 985-992, 1975

30.- Marshall, John
Futures and Option Contracting
Theory and Practice
South-Western Publishing Co., Cincinatti, Ohio, 1989

31.- Reiter, Stanley
A Market Adjustment Mechanism
Institute Paper Nº 1, Purdue University, Indiana, 1959

32.- Stahl, Dale y Fisher, Franklin
On Stability Analysis with Disequilibrium Awareness
Journal of Economic Theory, Vol.46, pp. 309-321, 1988