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Germán Coloma

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Bertrand and Price-Taking Equilibria in Markets with Product Differentiation

Germán Coloma [(*)]

Abstract

In this paper we show that a homogeneous-product market with multiple Bertrand equilibria becomes a market with a single Bertrand equilibrium when we introduce a small degree of product differentiation. When differentiation tends to zero, that Bertrand equilibrium converges to the unique price-taking equilibrium of the homogeneous-product market, which is in turn one of the multiple Bertrand equilibria for that market.

Resumen en castellano

En este trabajo se muestra que un mercado de un producto homogéneo que presenta múltiples equilibrios de Bertrand se convierte en un mercado con un único equilibrio de Bertrand cuando se le introduce un pequeño grado de diferenciación de productos. Cuando dicha diferenciación tiende a cero, el equilibrio de Bertrand converge al único equilibrio competitivo del mercado del producto homogéneo, que es a su vez uno de los múltiples equilibrios de Bertrand de este último mercado.

JEL Classification Number: D43, L13.

Keywords: Bertrand equilibrium, price-taking equilibrium, product differentiation.

1. Introduction

Since the very beginning of the history of the concept of Bertrand equilibrium (Bertrand, 1883), there exists the idea that such equilibrium exhibits some equivalence or convergence with the concept of perfectly competitive or price-taking equilibrium. However, Dastidar (1995) has shown that, in the context of oligopolies with homogeneous products and convex cost functions, Bertrand equilibria are typically multiple while price-taking equilibria are unique, and Vives (1999) has shown that, in those markets, the price-taking equilibrium allocation coincides with one of the possible Bertrand equilibrium ones[1]. These results are in sharp contrast with the ones that can be

(*) CEMA University; Av. Córdoba 374, Buenos Aires, C1054AAP, Argentina; Telephone: (54-11)6314-3000; E-mail: gcoloma@cema.edu.ar. I thank Alejandro Saporiti and Sergio Pernice for some comments to a preliminary version of this paper, which made me correct several mistakes. All remaining errors are imputable to me. The views and opinions expressed in this publication are those of the author and are not necessarily those of CEMA University.

[1] In previous work (Coloma and Saporiti, 2006) we have shown that some of those results can be extended to homogeneous-product markets with non-convex cost functions, which may have multiple Bertrand equilibria even in cases where no price-taking equilibria exist.

obtained for differentiated-product markets, in which Bertrand equilibria are typically unique (see, for example, Caplin and Nalebuff, 1991).

The aim of this paper is to develop a homogeneous-product model that follows Dastidar's idea, and to show that, if we allow for a small degree of product differentiation, it becomes a case where there is a single Bertrand equilibrium. When differentiation tends to zero, that Bertrand equilibrium converges to the unique price-taking equilibrium of the homogeneous-product market. The way to introduce product differentiation is to allow for a representative consumer who possesses a generalized CES utility function, for which the substitution among the different varieties of the same product can be measured through a single parameter. In order to keep the model more tractable, we will concentrate on a case with only two varieties, each of which is supplied by a different firm. The corresponding Bertrand equilibria, therefore, are those of a duopoly in which suppliers are symmetrical.

The paper is organized as follows. In section 2 we study the Bertrand equilibria of homogeneous-product markets, and find conditions for multiple Bertrand equilibria to exist. In section 3, we study the unique Bertrand equilibrium of the corresponding differentiated-product markets, and its convergence to the price-taking equilibrium outcome when differentiation tends to zero. Finally, in section 4, we analyze the main conclusions of the paper.

2. Homogeneous-product markets

Let us imagine a market with two firms, each of which with a continuous, differentiable, increasing and strictly convex total cost function $C(Q_i)$, where Q_i is the quantity supplied by the ith firm. Let us also assume that $C(0) = 0$.

The product traded in this market is homogeneous, with total demand equal to $Q = D(P)$, where Q is total quantity, P is the price paid by consumers, and D is a continuous, differentiable and decreasing function of P, with $\lim_{P\rightarrow\infty}D(P) = 0$.

In a situation of price competition, each of the two firms faces the following individual demand:

$$D_i(P_i, P_j) = \begin{cases} 0 & (\text{if } P_i > P_j) \\ \dfrac{D(P_i)}{2} & (\text{if } P_i = P_j) \\ D(P_i) & (\text{if } P_i < P_j) \end{cases} \quad ;$$

where P_i is the ith firm's price and P_j is the price chosen by its competitor[2].

The ith firm's profits, therefore, can be defined as:

$$\Pi_i(P_i, P_j) = P_i \cdot D_i(P_i, P_j) - C(D_i(P_i, P_j)) \quad ;$$

or, alternatively, as a function of its output, leaving implicit the price vector and the corresponding individual demand. This implies that:

$$\Pi_i(Q_i) = P_i \cdot Q_i - C(Q_i) .$$

Definition 1 (Price-taking equilibrium): *Given a non-negative price P_c, a price-taking equilibrium (PTE) is a pair $(Q_1, Q_2) \in \Re^2_+$ such that, for each $i = 1, 2$:*

$$Q_i = \arg\max_{Q_i \in \Re_+} \{P_c \cdot Q_i - C(Q_i)\} \qquad \text{(C1) ;}$$

$$P_c \cdot Q_i - C(Q_i) \geq 0 \qquad \text{(C2) ;}$$

$$Q_1 + Q_2 = D(P_c) \qquad \text{(C3) .}$$

Note that C3, together with the sharing rule implicit in the definition of individual demands, implies that, if (Q_1, Q_2) is a PTE for a given $P_c \geq 0$, then $Q_1 = Q_2 = D(P_c)/2$. We can therefore refer to (P_c, Q_i) as a PTE, understanding that this means that $(Q_1, Q_2) = (Q_i, Q_i)$ satisfies conditions C1-C3 above under the price P_c.

The assumptions about D and C guarantee that it is always possible to find a unique pair of positive values of P_c and Q_i that satisfies C1 and C3. C2, moreover, will also be satisfied by the pair (P_c, Q_i) implied by C1 and C3. Conceptually, this occurs because C1 and C3 determine an allocation for which price is equal to the marginal cost of each of the firms that operate in the market, and the strict convexity of $C(Q_i)$ guarantees that its average cost is always smaller than its marginal cost for positive values of Q_i. P_c, therefore, will always be larger than the corresponding average cost, and hence profits will be non-negative and C2 will be fulfilled in equilibrium.

Definition 2 (Bertrand equilibrium): *A pure-strategy Bertrand equilibrium (PBE) is a pair $(P_1, P_2) \in \Re^2_+$ such that, for each $i \neq j$:*

$$\Pi_i(P_i, P_j) \geq \Pi_i(\hat{P}, P_j) \quad (\text{for all } \hat{P} \in \Re_+) \qquad \text{(E1) ;}$$

$$\Pi_i(P_i, P_j) \geq 0 \qquad \text{(E2) ;}$$

$$Q_i(P_i, P_j) = D_i(\hat{P}, P_j) \qquad \text{(E3) ;}$$

[2] Note that this definition of the individual demand of the ith firm implies assuming an "equal sharing

where $Q_i(P_i, P_j)$ is the output supply of the ith firm at prices (P_i, P_j).

It is relatively easy to show that, if a PBE exists, then $P_1 = P_2 = P_b$. As the market-sharing rule assumed implies that $D_1(P_b, P_b) = D_2(P_b, P_b) = D(P_b)/2$, then E2 can be re-written as:

$$P_b \cdot \frac{D(P_b)}{2} - C\left(\frac{D(P_b)}{2}\right) \geq 0 \qquad \text{(E4) ;}$$

while E1 simply requires that:

$$P_b \cdot \frac{D(P_b)}{2} - C\left(\frac{D(P_b)}{2}\right) \geq \hat{P} \cdot D(\hat{P}) - C\left(D(\hat{P})\right) \quad (\text{for all } \hat{P} < P_b) \qquad \text{(E5) .}$$

When E4 is satisfied as a strict equality, we obtain the minimum price P_{min} that can be supported as a PBE. Similarly, when E5 is satisfied as a strict equality we get the maximum price P_{max} that can be supported as a PBE. For the set of PBE to be non-empty, it is necessary that $P_{max} \geq P_{min}$. In fact, if $P_{max} > P_{min}$, there exists a continuum of Bertrand equilibria (P_1, P_2), with the property that in each of them it holds that $P_1 = P_2 \in [P_{min}, P_{max}]$ [3]. One of the elements of this set is the PTE price (P_c), as it is shown in proposition 1.

Proposition 1: *If (P_c, Q_i) is a PTE, then (P_c, P_c) is a PBE, and $P_c \in [P_{min}, P_{max}]$.*

Proof: Assume, by contradiction, that (P_c, P_c) is not a PBE. Note first that, since (P_c, Q_i) is a PTE, then C2 and C3 imply that E3 and E4 are satisfied at (P_c, P_c). Hence, there must exist a price P_i such that $\Pi_i(P_i, P_c) > \Pi_i(P_c, P_c) \geq 0$. That means that $\Pi_i(P_i, P_c) = P_i \cdot D_i(P_i, P_c) - C(D_i(P_i, P_c)) > 0$ and, therefore, $P_i < P_c$.
In equilibrium, then, $Q_i = D_i(P_i, P_c) = D(P_i)$. Totally differentiating $P_i \cdot Q_i - C(Q_i) > 0$ with respect to Q_i, we have $P_i > \partial C(Q_i)/\partial Q_i$. But we know, by C1, that $P_c = \partial C(Q_c)/\partial Q_i$. Moreover, since $P_i < P_c$, $D' < 0$ and $C'' > 0$, it follows that $\partial C(Q_c)/\partial Q_i < \partial C(Q_i)/\partial Q_i$. Therefore, $P_i > \partial C(Q_i)/\partial Q_i$ implies that $P_i > P_c$, and this is a contradiction. Hence, (P_c, P_c) is a PBE, i.e., $P_c \in [P_{min}, P_{max}]$, qed[4].

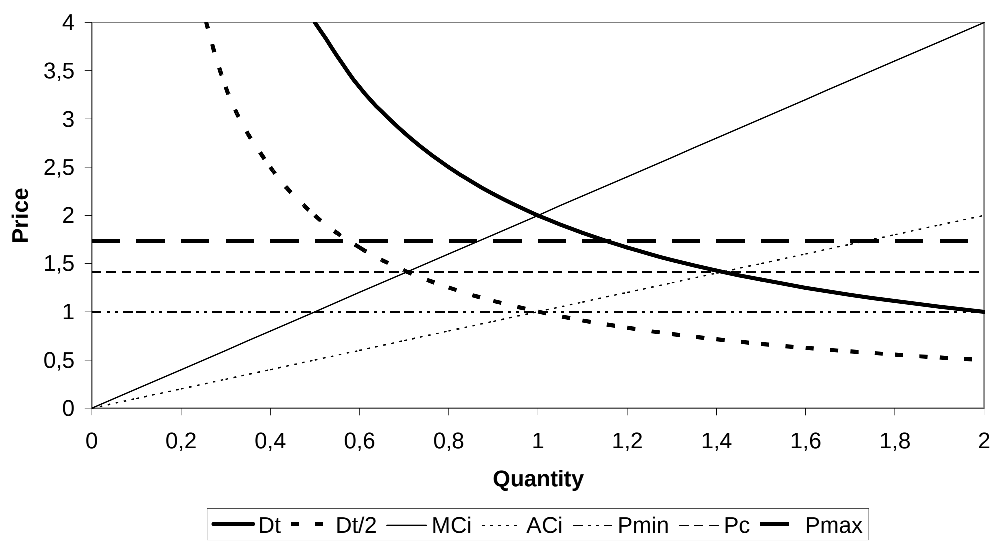
The existence of the price-taking equilibrium is therefore a sufficient condition for the Bertrand equilibria to exist, and the reverse is also true. This is because, by definition, $P_c > P_{min}$ and $P_{max} > P_c$, so if the interval $[P_{min}, P_{max}]$ is not empty, then P_c

rule". For other alternative rules applicable to situations of price competition, see Hoernig (2007).
[3] Note that all these are "Bertand equilibria" and not "Bertrand-Edgeworth equilibria", since E3 requires that firms meet all the demand at the equilibrium prices. Their only strategic choice, therefore, is the price that they charge and not the quantity that they sell. For an explanation of the difference between Bertrand and Bertrand-Edgeworth equilibria, see Vives (1999), chapter 5.

must belong to that interval and the PTE must therefore exist. The PTE allocation, therefore, is one of the multiple PBE allocations that this type of markets exhibit.

Let us now consider a numerical example of a market with two firms, each of which with a total cost function $C_i = Q_i^2$. Let us assume that the total demand function of this market is $Q = 2/P$. The corresponding PTE occurs when $P_c = 2 \cdot Q_i$ (profit maximization condition for the ith firm) and the market-clearing price is equal to $P_c = 2/(2 \cdot Q_i)$. Equating both conditions we end up with an equilibrium in which $Q_i = 0.7071$ and $P_c = 1.4142$. This allocation generates a positive profit for the two duopolists, which is equal to $\Pi_i = 0.5$. To check that P_c is a price that belongs to the interval of PBE symmetric equilibria, we can check that $P_{max} > P_c > P_{min}$. Indeed, in this example $P = P_{min}$ is the number for which it simultaneosuly holds that $P = Q_i$ and $P = 2/(2 \cdot Q_i)$, and this occurs when $Q_i = 1$ and $P_{min} = 1$. Conversely, $P = P_{max}$ is the number for which it simultaneosuly holds that $P \cdot Q_i - Q_i^2 = P \cdot (2 \cdot Q_i) - (2 \cdot Q_i)^2$ and $P = 2/(2 \cdot Q_i)$, and this occurs when $Q_i = 0.5774$ and $P_{max} = 1.7321$. As we see, $1.7321 > 1.4142 > 1$, and this confirms that the interval $[P_{max}, P_{min}]$ is not empty and that P_c belongs to that interval.

1. Homogeneous product market



The numerical example referred to in the previous paragraphs is graphically represented on figure 1. In it we see the total demand curve that we have postulated (*Dt*) and the portion of that curve that corresponds to each of the two firms that operate in the

[4] This proof is a slight variation of a similar one that appears in Coloma and Saporiti (2006).

market (*Dt/2*). We have also depicted the individual marginal cost curve (*MCi*), and the individual average cost curve (*ACi*). Given those elements, the lower limit of the interval of PBE prices (*Pmin*) is determined by the point where *ACi* crosses *Dt/2*, while the PTE price (*Pc*) is that for which *MCi* crosses *Dt/2*. Finally, the upper limit of the interval of PBE prices (*Pmax*) is that for which the distance between *MCi* and *Dt/2* exactly coincides with the distance between *Dt* and *MCi*.

3. Differentiated-product markets

Let us now assume that each of the duopolists that operate in the market supplies a differentiated product. For the sake of simplicity, we will assume that differentiation is symmetric, and each firm faces the following demand function:

$$Q_i = \frac{D(R_i)}{2} \qquad \text{(for i = 1, 2)} \quad .$$

In this context, *D* is the same demand function used in section 2, and R_i is the following function of P_i and P_j:

$$R_i = \frac{P_i + P_i^{\frac{1}{\theta}} \cdot P_j^{\frac{\theta-1}{\theta}}}{2} \qquad \text{(for i = 1, 2 and j} \neq \text{i)} \ ;$$

where $\theta \in [0, 1]$ is a parameter that measures product differentiation. When θ tends to one, product differentiation is maximal, and $Q_i = D(P_i)/2$ for any value of P_i and P_j. When θ tends to zero, conversely, the product approaches homogeneity (and the individual demand function converges to the one that we have seen in section 2).

This demand function can be derived from the optimization problem of a representative consumer. The preferences of this representative consumer are a generalization of the so-called "constant-elasticity-of-substitution utility function" (CES), whose form is $U = U(Q_1^{1-\theta}+Q_2^{1-\theta}, Q_3, ..., Q_n)$. The two products under analysis are products 1 and 2, respectively, and the implicit assumption is that income and prices of the other products are held constant.

In a market like this, a PBE must fulfill the same conditions stated by definition 2, namely:

$$\Pi_i(P_i, P_j) \geq \Pi_i(\hat{P}, P_j) \quad \text{(for all } \hat{P} \in \Re_+) \qquad \text{(E1)} \ ;$$

$$\Pi_i(P_i, P_j) \geq 0 \qquad \text{(E2)} \ ;$$

$$Q_i(P_i, P_j) = D_i(\hat{P}, P_j) \qquad \text{(E3)} .$$

If a PBE exists for this market, it must be unique. It will imply an allocation formed by a symmetric pair of prices and quantities "P_i, Q_i" that simultaneously satisfies conditions E1 and E3, and it will exist as long as that pair also satisfies condition E2. When θ tends to zero, this allocation converges to the PTE of the homogeneous-product case, and it therefore exists. It also exists for any $\theta > 0$. Bertrand equilibrium prices are increasing in θ, and the profits that the firms obtain in equilibrium are also increasing in θ. The PTE allocation, conversely, is the same for any $\theta \in [0, 1]$, and therefore its corresponding equilibrium price is always smaller than the Bertrand equilibrium price. All these results are more formally stated in propositions 2 and 3.

Proposition 2: *If (P_c, Q_i) is a PTE when $\theta = 0$, then it is also a PTE for all $\theta > 0$.*
Proof: If (P_c, Q_i) is a PTE when $\theta = 0$, then $P_c = \partial C_i/\partial Q_i$ and $Q_i = D(P_c)/2$. As this implies a symmetric allocation, then $P_c = P_i = R_i$, and therefore $R_i = \partial C_i/\partial Q_i$ and $Q_i = D(R_i)/2$. As these equalities hold for any $\theta > 0$, then (P_c, Q_i) is also a PTE for any $\theta > 0$, qed.

Proposition 3: *When $\theta > 0$, there exists a unique PBE allocation, whose equilibrium price converges to the PTE price ($P_i \to P_c$) when $\theta \to 0$.*
Proof: Applying E1 and E3 in a context of a continuous, decreasing and differentiable demand function and continuous, increasing, differentiable and convex cost functions, implies that the ith firm maximizes its profits when it holds that:

$$\frac{\partial \Pi_i}{\partial P_i} = Q_i + \left(P_i - \frac{\partial C_i}{\partial Q_i}\right) \cdot \frac{(\partial D/\partial R_i)}{2} \cdot \left(\frac{1+\theta}{2\cdot\theta}\right) = 0 \qquad \Rightarrow \qquad \frac{P_i - \partial C_i/\partial Q_i}{P_i} = \frac{2\cdot\theta}{\eta\cdot(1+\theta)} \quad ;$$

where η is the absolute value of the own-price elasticity of $D(P_i)$. If $\theta > 0$, then $(2\cdot\theta)/[\eta\cdot(1+\theta)] > 0$, and therefore $P_i > \partial C(Q_i)/\partial Q_i > \partial C(Q_c)/\partial Q_i = P_c$. This guarantees that E2 is satisfied. When $\theta \to 0$, then $(2\cdot\theta)/[\eta\cdot(1+\theta)] \to 0$, and therefore $P_i \to \partial C(Q_i)/\partial Q_i \to \partial C(Q_c)/\partial Q_i = P_c$, qed.

To illustrate these results, consider the numerical example developed in section 2. If we introduce product differentiation in this numerical example, then the demand function for the ith firm becomes:

$$Q_i = \frac{4}{P_i + P_i^{\frac{1}{\theta}} \cdot P_j^{\frac{\theta-1}{\theta}}} \qquad ;$$

and the symmetric PTE occurs when $Q_i = 0.7071$ and $P_c = 1.4142$. This is independent of the value of θ, because, when $P_i = P_j$, then $Q_i = 2/P_i$. The corresponding PBE allocation, conversely, occurs when $P_i = [2 \cdot (1+\theta)/(1-\theta)]^{0,5}$ and $Q_i = [(1-\theta)/(2+2 \cdot \theta)]^{0,5}$, and does therefore depend on the value of θ. The PBE price is increasing in the parameter θ, and converges to infinity when θ tends to one and to $P_i = P_c = 1.4142$ when it tends to zero. The PBE quantity is decreasing in the parameter θ, and it converges to zero when θ tends to one and to $Q_i = 0.7071$ when θ tends to zero.

2. PBE and PTE prices with product differentiation

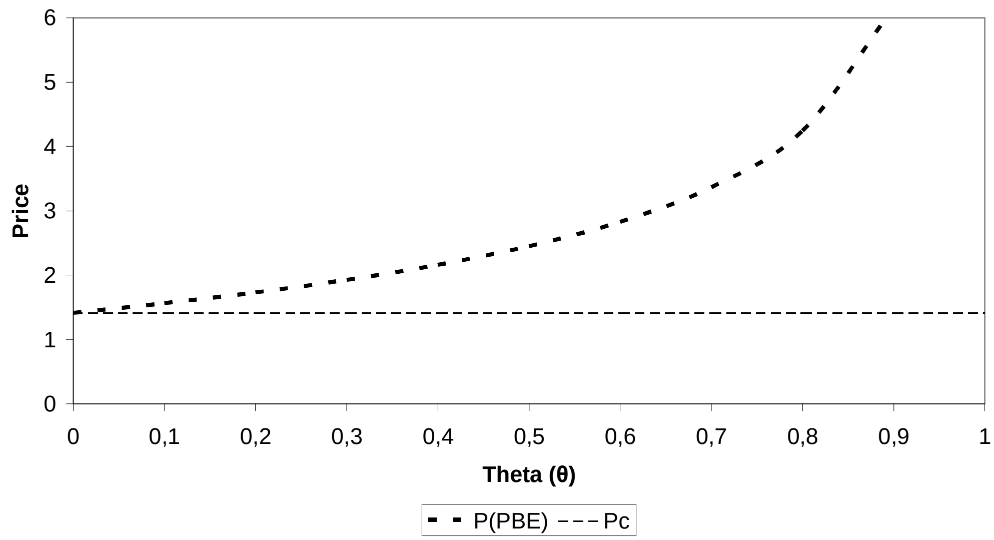


Figure 2 is a graphical representation of our numerical example. In it we can see that, while the PTE price (*Pc*) is always the same for any value of θ between 0 and 1, the PBE price (*P(PBE)*) is increasing in θ, and it equals the PTE price when θ converges to zero.

The reader may wonder why the range of multiple symmetric PBE of the homogeneous-product case, given by $[P_{min}, P_{max}]$, disappears when product differentiation arises. The answer has to do with the fact that, when $\theta = 0$, individual demands are not continuous in the symmetric equilibria, and they "jump" from Q_i to $2 \cdot Q_i$ when P_i decreases slightly. If $\theta > 0$, conversely, individual demands are continuous when both firms charge the same price, and this continuity is precisely the characteristic that determines that the PBE allocation is unique.

4. Concluding remarks

This paper has tried to conciliate two opposing results of the literature associated to the concept of Bertrand equilibrium. One of them is the one that appears in Dastidar (1995), who shows that pure-strategy Bertrand equilibria are typically multiple in a homogeneous-product case[5]. The other one is the one that appears in Caplin and Nalebuff (1991) and other similar articles, which show that Bertrand equilibrium is typically unique in a differentiated-product case.

By building a duopoly model in which product differentiation is measured through a single parameter, we find that uniqueness is preserved, and that the Bertrand equilibrium of a differentiated-product market converges to the price-taking equilibrium when differentiation tends to zero. Consequently, the PTE allocation is the only PBE allocation of the homogeneous-product case that survives a "perturbation" consisting in the introduction of a small degree of product differentiation.

References

Bertrand, Joseph (1883). "Théorie Mathématique de la Richesse Social"; *Journal des Savants*, vol 68, pp 499-508.

Caplin, Andrew and Barry Nalebuff (1991). "Aggregation and Imperfect Competition: On the Existence of Equilibrium"; *Econometrica*, vol 59, pp 25-59.

Coloma, Germán and Alejandro Saporiti (2006). "Bertrand Equilibria in Markets with Fixed Costs"; Economics Discussion Paper 0627, University of Manchester.

Dastidar, Krishnendu (1995). "On the Existence of Pure Strategy Bertrand Equilibrium"; *Economic Theory*, vol 5, pp 19-32.

Hoernig, Steffen (2002). "Mixed Bertrand Equilibria Under Decreasing Returns to Scale: An Embarrassment of Riches"; *Economics Letters*, vol 74, pp 359-362.

Hoernig, Steffen (2007). "Bertrand Games and Sharing Rules"; *Economic Theory*, vol 31, pp 573-585.

Vives, Xavier (1999). *Oligopoly Pricing*. Cambridge, MIT Press.

[5] See also Hoernig (2002), who shows that in that context there are also multiple mixed-strategy Bertrand equilibria.